Materials Science-Poland, Vol. 25, No. 2, 2007

Fracton oscillations in the net fractals

Z. BĄK*

Institute of Physics, Jan Długosz University of Częstochowa,
al. Armii Krajowej 13/15, 42-201 Częstochowa, Poland

Assuming that the force constants scale as $\sigma(\lambda x) = \lambda^{-\alpha}\sigma(x)$, we construct the model of elastic (linear) excitations on a fractal (fractons). We show that the fractons, a specific class of fractals, "net fractals", can be assumed as the log-scale phonons. Further, we discuss propagation of elastic medium perturbation in terms of fractional dynamics. Within this approach we find two different modes of vibrations, both expressed with the help of generalized Mittag–Leffler functions.

Key words: *fractals; fractons; fractional dynamics*

1. Introduction

The concept of fractal has become a powerful tool in analysis of common aspects of many complex processes observed in physics, biology, chemistry or earth sciences. Brownian motion, turbulence, colloid aggregation or biological pattern formation can be fully understood only when the idea of self-similarity of fractal structures is applied [1]. The hallmark of fractality is a hierarchical organization of its elements, described by discrete scaling laws, which makes the fractal, regardless of the magnification or contraction scale, look the same. This property of fractals is called self-similarity, self-affinity or self-replicability. Although physical systems modelled by fractals are non-translation invariant, it is a well known fact that the self-similar fractals, as well as the physical quantities on fractal systems, show log-periodicities [1]. This opens the possibility to describe the symmetries of self-similar fractals in the way that is reminiscent of conventional formalism developed for crystalline systems. Motivated by this fact, we present a study of fractal excitations (fractons), which is similar in spirit to the phonon approach in the solid state theory.

A self-similar symmetry of a fractal is a transformation that leaves the system invariant, in the sense that taken as a whole it looks the same after transformation as it

*E-mail: z.bak@ajd.czest.pl

did before, although individual points of the pattern may be moved by the transformation. We say that $\mathbf{K} \subset \mathbf{R}^n$ satisfies the scaling law **S**, or is a self-similar fractal, if $\mathbf{S}:\mathbf{K} = \mathbf{K}$. Let us limit our considerations to fractals in which the self-similarity can be realized only via linear maps, i.e., transformations which point $\mathbf{r} = (x_1, x_2, x_3) \in \mathbf{K} \subset \mathbf{R}^3$ transform into point $\mathbf{r}' = (x_1'\ x_2', x_3')$ according to the formula $x_i' = S_{i1} x_1 + S_{i2} x_2 + S_{i3} x_3$; where $i = 1, 2, 3$. The vector form of the linear self-similar transformation can be written as $\mathbf{r}' = \mathbf{S}:\mathbf{r}$, where **S** is the matrix of the linear self-similar transformation. If we orient coordinate axes along the eigenvectors of the matrix **S** (i.e., $\mathbf{x} = (x_1, x_2, x_3) \to (\varepsilon, \eta, \rho)$), then the linear self-similar mapping reduces to the transformation $\mathbf{S}$: $(\varepsilon, \eta, \rho) \to (\lambda_1\varepsilon, \lambda_2\eta, \lambda_3\rho)$. In the case of infinite-size fractals, also the inverse $\mathbf{S}^{-1}$ mapping fulfils the self-similarity conditions $\mathbf{S}^{-1}:\mathbf{K} = \mathbf{K}$ and for any $\mathbf{x} \in \mathbf{K}$, we have

$$\mathbf{S}^{-1} : \mathbf{x} = \mathbf{S}_1^{-1} \cdot \mathbf{S}_2^{-1} \cdot \mathbf{S}_3^{-1} : \mathbf{x} = \left(\lambda_1^{-1}\varepsilon, \lambda_2^{-1}\eta, \lambda_3^{-1}\rho\right) \tag{1}$$

Consider a more general transformation of the type $S^{(m,n,l)} = (S_1)^n \cdot (S_2)^m \cdot (S_3)^l$, where $(S_i)^n$ denotes n-tuple superposition of transformation S_i and define a class of infinite "net fractals" G_{nf}, for which the relation $S^{(m,n,l)}:G_{nf} = G_{nf}$ is valid. Action of $S^{(m,n,l)}$ transforms any point $x \in \mathbf{R}^3$ according to the formula $S^{(m,n,l)}:x = (\lambda_1^{-1}\varepsilon, \lambda_2^{-1}\eta, \lambda_3^{-1}\rho)$, where m, n, l are arbitrary (negative or positive) integers. In view of this relation, we have that $S^{(m,n,l)}:G_{nf} \subset G_{nf}$, i.e. $S^{(m,n,l)}$ are the injective scaling mappings. For any linear S_1 and $F_1 \subset \mathbf{R}$ by definition we have $S_1:F_1 = F_1$ and for any $x_0 \in F_1$ we have S_1: $x_0 = \lambda_1 x_0$, consequently $(S_1)^m$: $x_0 = \lambda_1^m: x_0$. Using the logarithmic scale we have $\log(x_m/x_0) = m\ln\lambda_1 (m = \pm1, \pm2\ldots)$. This is nothing but a 1 D crystal lattice with the lattice spacing given by $a_1 = \ln\lambda_1$. Using the multi-logarithmic scale, we can see that the family of mappings $S^{(m,n,l)}$ is isomorphic with a 3D crystal lattice. This means that the isomorphism $S^{(m,n,l)} \leftrightarrow (ma_1, na_2, la_3)$ holds, the same refers to the placement of its characteristic building blocks. The purpose of this paper is to study the vibrations in a system deformable over fractal subset.

2. Fractons

Most theoretical studies of the vibrations of a fractal limit to considerations at universal level without referring the specific physical model. In our study, we focus on a specific model which, we believe, describes behaviour of some real systems. Consider a "net fractal" cluster as defined above, consisting of N atoms with unit mass and linear springs connecting nearest-neighbour sites. The equations of motion of the atoms are [2]:

$$\ddot{u}_n(t) + \sum_m k_{n,m} u_n(t) = 0 \tag{2}$$

where the sum goes over all nearest neighbour sites of the fractal site n. When trying to work with Eq. (2), one meets two problems: first, elastic constants $k_{m,n}$ and mass distribution depend on coordinates, and secondly, it is associated with the ambiguities in the definitions of the local displacements u_n. The local strain **e** ($\mathbf{e} \propto \nabla \mathbf{u}(r,t)$) and local displacements on the fractal system can be defined in two ways. One of the definitions refers to the internal geometry and microscopic interactions, while the other defines the strain directly in terms of the effect of deformations on the (suitably averaged) mass distribution. Since we are interested in the study of fractal acoustics, we should use the latter definition which is directly relevant to the experiment. As was pointed out by Alexander [2], in this case the vibrational displacements are the vectors in the embedding space and are not restricted by the internal geometry of the fractal. Let us now discuss the non-homogeneities of mass and force constants. Since, due to rapid fluctuations, on a short length scales the strains and density can be defined only as the scale dependent local averages [2], we can assume that fractal of the size r has, on the average a mass $m(r) = m_0(r/a)^d$, where d is the fractal mass dimension. It is natural to assume that the self-similarity of the fractal is reflected also in the dilation symmetry. Assuming that ω is the eigenfrequency of the fracton oscillations, we can find that the force constants k_i scale as $k_i = m_k \omega_k^2 \propto (r/a)^d \omega_\kappa^2$. In view of the latter relation, from here on, we assume that the forces which tend to restore the equilibrium positions of species, are linear (as regards to the coordinates of the excited fractal system). However, contrary to the conventional solid elastic constants, they are not homogenous and depend on coordinates.

Let us assume that elastic forces follow the common power law scaling with the separation [2]. As we have shown above, when presented in the logarithmic coordinates, the mass density of such a fractal becomes uniform, the same refers to the elastic constants. Suppose the fractal is perturbed locally (e.g., in the vicinity of the equilibrium position x_0, with the energy ε_0 and consider the amplitude of this excitation. In real space, the amplitude of local fluctuation has the form $u_n = |x_{0n} - x_n|$, while in the log coordinates we have $\zeta_n = |\xi_n^0 - \xi_n|$, where $\zeta_n = \ln x_n$. Consider first a somewhat unrealistic case when there are no broken bonds in the log-scale picture. In this case (in the log scale) we have a homogeneous system with uniform mass and elastic constant distribution. Under above conditions, application of the continuous medium approximation is justified. Thus, when perturbed the log coordinates ζ_n and the local displacement $\zeta_n(x,t)$ should satisfy the classical wave equation $\nabla^2 \zeta - (1/c^2)(\partial^2 \zeta/\partial t^2) = 0$ with the plane wave solution $\zeta_i = \zeta_i^0 \exp(ik_i\zeta_i - i\omega t) = \zeta_i^0 (x_i)^{ik_i} \exp i\omega t$, when the relation $\zeta_n = \ln x_n$ is taken into account. As we can see from above, the fracton appears to be the log-scale phonon. When transformed to the physical space, the log-scale phonon solution displays power law scaling with purely imaginary scaling exponent. The extensive discussion of the systems with complex scaling factors was given by Sornette [3], who proved that this type of scaling results in log-periodic oscillations of physical quantities. In classical physics, the Huygens principle says that any point of an isotropic medium reached by a travelling wave is the source of an outcoming spherical wave. By analogy to the Huygens principle, we assume that the fractal excitation can excite neighbouring fractal cluster provided that there is some contact between them.

This means that wave propagation in a fractal, granular medium is associated with chaotic trajectories, which travel over chaotically distributed fractal clusters. There is no straight lanes which characterize sound propagation in conventional physics. The dead ends with no connection to the neighbouring fractal clusters play the role of temporary traps which slow down the propagation. This is nothing but the picture of continuous random walks (CTRW). Such scenario indicates that the model of space-time fractional diffusion can be applied in description of wave propagation [5] . Within this approach, the generalized wave equation for the amplitude of local oscillation $u(x,t)$ can be formally written as [6]

$$ {}_{|x|}D^{2\beta}u(x,t) - \frac{1}{c^{2\alpha}}\,{}_{t}D_{*}^{2\alpha}u(x,t) = 0 \tag{3} $$

The fractional time derivative reflects the CTRW effect while fractional space derivatives describe the reduced dimensionality of the system. Let us write down the fractal counterpart of the wave-equation. Since there are many definitions of fractional derivatives in any approach which involves the fractional calculus techniques one should define which definition of fractional pseudo-differential are used. Following the approaches of [4, 5] we assume that the fractional time derivatives ${}_{t}D^{\alpha}{}_{*}$ are that of Caputo, while the space ${}_{|x|}D^{\beta}$ ones are these of Riesz [5, 6]. To avoid many cumbersome aspects of these operators it is enough to work with the Fourier transforms $F\{u\}$ of the oscillation amplitude $u(x,t)$ [5]:

$$ F\{{}_{|x|}D^{2\beta}u(x,t)\} = -|k|^{2\beta}F\{u(x,t)\} \tag{4} $$

where k is the transform variable. A similar relation holds for the time derivative. Finding the solution of Eq. (3) for arbitrary values α and β in its most general form is impossible, however, under additional assumptions we can find some specific solutions. Indeed, suppose we can separate the variables x and t. This means we assume that $u(x,t) = u_1(x)u_2(t)$. Provided that our solutions fulfill such an assumption, we can rewrite Eq. (3) as:

$$ \frac{1}{u_1(x)}\,{}_{|x|}D^{2\beta}u_1(x) - \frac{1}{c^{2\alpha}u_2(t)}\,{}_{t}D_{*}^{2\alpha}u_2(t) = 0 \tag{5} $$

Equation (4) is equivalent to two independent differential equations of single variable x or t which solved give us the oscillation amplitude in the form

$$ u(x,t) = u_o|x|^{\beta-1}t^{\alpha-1}E_{\beta,\beta}(iK|x|)E_{\alpha,\alpha}(iKct) \tag{6} $$

where $E_{\alpha,\alpha}(x)$ is the generalized Mittag–Leffler function given by:

$$ E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j\alpha+\beta)} $$

3. Discussion and summary

The Mittag–Leffler function $E_{\alpha,\alpha}(x)$ behaves like a stretched-exponential $\exp(-|x|^{\alpha})$, at short times [4, 5] and like x^{α} as $x \to \infty$. As we know, the generalized Mittag–Leffler function can be assumed as the counterparts of exponential function defined on a space of fractional dimension. This means that for $\alpha \to 1$ solution of Eq. (6) reduces to conventional plane-wave $u \propto \exp(kx - i\omega t)$. Suppose that our finite system is extended over a continuous manifold M limited by the boundary ∂M. Let us set the typical, Cauchy type, boundary conditions in the form $u(x,t)|_{\partial M} = 0$. It can be easily seen that in the case of symmetrical M (e.g., $x \in [-L, L]$) solution of Eq. (6) can satisfy these boundary conditions. Indeed, our boundary conditions are equivalent to $E_{\alpha,\alpha}(iK|L|) = 0$. This means that the number of allowed vibrational eigenmodes is equal to the number of zeroes x_n, i.e. $E_{\alpha,\alpha}(x_n) = 0$ of the generalized Mittag–Leffler functions. Thus, the allowed values of K (K is the counterpart of the wave vector k in the conventional, bulk systems) becomes quantized, $K_n = x_n/L$. As we know, Mittag–Leffler functions have a finite and odd number of zeroes thus the conditions $u(x,t)|_{\partial M} = 0$ makes that only a finite number of vibrational eigenmodes within finite fractal system is possible.

In summary, we have shown that the fracton excitations on the “net fractals”, when presented in the log-scale, resemble phonons in conventional 3 D solids. The results obtained above refer to the “net fractals” which are ideal generalizations of some real fractals. Most real fractals consist of backbone and sidebranches (dead ends) attached to it. Thus, real fractals differ from the discussed above “net fractals” since they show the log-periodicity, at least along the backbone. It was suggested [2] that during vibration only the fractal backbone is stressed, while these parts of the mass which are located on the sidebranches are moved along rigidly without being strained. The arguments given above suggest that in a real dendritic fractal at most one vibrational eigenmode should display the log-scale features. We should point here that the fractal systems can be formed by an assembly of mobile particles (e.g. excitons or electrons confined within quantum well [7]). Their mobility and vibrations described by fractional spectral dimension are the source of unusual physical phenomena in these systems [8]. We believe that the presented model of fractal excitations provides a guideline for analysis of other phenomena on mesoscopic fractal systems like spin waves, resonant transmission/absorption through fractal slits, fractal plasmons or fractal antennas (fractal electrodynamics).

References

[1] VICSEK T., SHLESINGER M., MATSUSHITA M., *Fractals in Natural Sciences*, World Scientific, Singapore, 1994.
[2] ALEXANDER S., Phys. Rev. B, 40 (1989), 7953.
[3] SORNETTE D., Phys. Rep., 297 (1998), 239.

[4] SAMKO S.G., KILBAS A.A., MARICHEV O.I., *Integrals and Derivatives of Nonintegral Order with Some Applications* (in Russian), Nauka i Tekhnika, Minsk, 1987.
[5] MAINARDI F., LUCHKO Y., PAGNANI G., Fract. Calc. Appl. Anal., 4 (2001), 153.
[6] HANYGA A., Proc. Roy. Soc. (London), A, 457 (2001), 2993.
[7] POVSTENKO Y.Z., Int. J. Eng. Sci., 43 (2005), 977.
[8] MIKHAILOV I.D., BETANCUR F.J., ESCORCIA R.A., SIERRA-ORTEGA J., Phys. Rev. B, 67 (2003), 115317.
[9] BAK Z., Phys. Rev. B, 68 (2003), 064511.

Received 7 May 2006
Revised 1 September 2006